

An integral geometry lemma and its applications: the nonlocality of the Pavlov equation and a tomographic problem with opaque parabolic objects

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Abstract

As in the case of soliton PDEs in 2+1 dimensions, the evolutionary form of integrable dispersionless multidimensional PDEs is non-local, and the proper choice of integration constants should be the one dictated by the associated Inverse Scattering Transform (IST). Using the recently made rigorous IST for vector fields associated with the so-called Pavlov equation $v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0$, we have recently established that, in the nonlocal part of its evolutionary form $v_t = v_x v_y - \partial_x^{-1} \partial_y [v_y + v_x^2]$, the formal integral ∂_x^{-1} corresponding to the solutions of the Cauchy problem constructed by such an IST is the asymmetric integral $-\int_x^\infty dx'$. In this paper we show that this results could be guessed in a simple way using a, to the best of our knowledge, novel integral geometry lemma. Such a lemma establishes that it is possible to express the integral of a fairly general and smooth function $f(X, Y)$ over a parabola of the (X, Y) plane in terms of the integrals of $f(X, Y)$ over all straight lines non intersecting the parabola. A similar result, in which the parabola is replaced by the circle, is already known in the literature and finds applications in tomography. Indeed, in a two-dimensional

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linear tomographic problem with a convex opaque obstacle, only the integrals along the straight lines non-intersecting the obstacle are known, and in the class of potentials $f(X, Y)$ with polynomial decay we do not have unique solvability of the inverse problem anymore. Therefore, for the problem with an obstacle, it is natural not to try to reconstruct the complete potential, but only some integral characteristics like the integral over the boundary of the obstacle. Due to the above two lemmas, this can be done, at the moment, for opaque bodies having as boundary a parabola and a circle (or, more generally, an ellipse). We expect that this result can be extended to a larger class of convex opaque bodies.

1 Introduction

Integrable dispersionless PDEs in multidimensions, intensively studied in the recent literature (see [1] for an account of the vast literature on this subject), arise as the condition of commutation $[L, M] = 0$ of pairs of one-parameter families of vector fields. A novel Inverse Scattering Transform (IST) for vector fields has been constructed, at a formal level in [2], [3], [4], [5], to solve their Cauchy problem, obtain the long-time asymptotics, and establish if, due to the lack of dispersion, the nonlinearity is strong enough to cause a gradient catastrophe at finite time [6], [7]. Due to the novel features of such IST (the corresponding operators are unbounded, the kernel space is a ring, the inverse problem is intrinsically non-linear), together with the lack of explicit regular localized solutions, it was important to make this IST rigorous, and this goal was recently achieved on the illustrative example of the so-called Pavlov equation [8], [9], [10]

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \quad v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \quad (1.1)$$

arising in the study of integrable hydrodynamic chains [8], and in Differential Geometry as a particular example of Einstein - Weyl metric [10]. It was first derived in [11] as a conformal symmetry of the second heavenly equation.

In the form (1.1) it is not an evolution equation. To rewrite it in the evolution form, we have to integrate it with respect to x :

$$v_t = v_x v_y - \partial_x^{-1} \partial_y [v_y + v_x^2], \quad v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \quad (1.2)$$

where ∂_x^{-1} is the formal inverse of ∂_x . Of course, it is defined up to an arbitrary integration constant, depending on y and t . On the other hand, the IST for integrable dispersionless PDEs provides us with a unique solution of the Cauchy problem

in which the function $v(x, y, 0)$ is assigned, corresponding to a specific choice of such integration constant.

In a recent paper we have specified the choice of the integration constant. More precisely, we have shown that the IST formalism for the Pavlov equation, a nonlinear analogue of the direct and inverse Radon Transform [5], corresponds to the following evolutionary form of the Pavlov equation:

$$v_t(x, y, t) = v_x(x, y, t) v_y(x, y, t) + \int_x^{+\infty} [v_y(x', y, t) + (v_{x'}(x', y, t))^2]_y dx', \quad t \geq 0. \quad (1.3)$$

In addition, for any smooth compact support initial condition and any $t > 0$, the solution develops the constraint

$$\partial_y \mathcal{M}(y, t) \equiv 0, \quad \text{where} \quad \mathcal{M}(y, t) = \int_{-\infty}^{+\infty} [v_y(x, y, t) + (v_x(x, y, t))^2] dx. \quad (1.4)$$

identically in y and t , but, unlike the Manakov constraints for the Kadomtsev-Petviashvili (KP) [12] and for the dispersionless Kadomtsev-Petviashvili (dKP) [13, 14, 15] equations, no rapidly decaying smooth initial data can satisfy this condition at $t = 0$.

We remark that the problem of non-locality is not typical of integrable dispersionless PDEs only, but it is also a generic feature of soliton PDEs with 2 spatial variables. Therefore the problem of choosing proper integration constants is very important also in the soliton context, and the IST provides the natural choice. This problem was first posed and discussed in [16] for the KP equation. The final answer for KP was obtained in [17], and, later, in [18].

In the remaining part of this introduction we summarize the basic formulas of the IST for the Pavlov equation (see, for instance, [19]) that will be used in this paper.

1.1 Summary of the IST for the Pavlov equation

The Pavlov equation is the commutativity condition $[L, M] = 0$ for the following pair of vector fields:

$$\begin{aligned} L &\equiv \partial_y + (\lambda + v_x) \partial_x, \\ M &\equiv \partial_t + (\lambda^2 + \lambda v_x - v_y) \partial_x. \end{aligned} \quad (1.5)$$

Assuming, as in [19], that the Cauchy datum $v(x, y, 0)$ has compact support

$$v(x, y, 0) = 0 \quad \text{if} \quad |x| > D_x \quad \text{or} \quad |y| > D_y, \quad (1.6)$$

is smooth and satisfies some small norm conditions, we define the spectral data using the following procedure:

1. We define the real Jost eigenfunctions $\varphi_{\pm}(x, y, \lambda)$, $\lambda \in \mathbb{R}$ as the solutions of the equation

$$L\varphi_{\pm}(x, y, \lambda) = 0,$$

with the boundary condition:

$$\varphi_{\pm}(x, y, \lambda) \rightarrow x - \lambda y \text{ as } y \rightarrow \pm\infty,$$

using the correspondent vector fields ODE:

$$\frac{dx}{dy} = \lambda + v_x(x, y) \quad (1.7)$$

2. If we denote by $x_-(y, \tau, \lambda)$ the solution of (1.7) with the following asymptotics:

$$x_-(y, \tau, \lambda) = \tau + \lambda y + o(1) \text{ as } y \rightarrow -\infty,$$

then the classical time-scattering datum $\sigma(\tau, \lambda)$ is defined through the following formula:

$$\sigma(\tau, \lambda) = \lim_{y \rightarrow +\infty} [x_-(y, \tau, \lambda) - \tau - \lambda y]. \quad (1.8)$$

Equivalently,

$$\sigma(\tau, \lambda) = \int_{-\infty}^{\infty} v_x(x_-(y, \tau, \lambda), y) dy.$$

In the linear limit $v \ll 1$ the scattering datum $\sigma(\tau, \lambda)$ reduces to the Radon transform of $v_x(x, y)$ [6].

3. The spectral data $\chi_{\pm}(\tau, \lambda)$ are defined as the solutions of the following shifted Riemann-Hilbert (RH) problem:

$$\sigma(\tau, \lambda) + \chi_+(\tau + \sigma(\tau, \lambda), \lambda) - \chi_-(\tau, \lambda) = 0, \quad \tau, \lambda \in \mathbb{R}, \quad (1.9)$$

where $\chi_{\pm}(\tau, \lambda)$ are analytic in τ in the upper and lower half-planes \mathbb{C}^{\pm} respectively, and

$$\chi_{\pm}(\tau, \lambda) \rightarrow 0 \text{ as } |\tau| \rightarrow \infty.$$

If the potential $v(x, y, t)$ evolves in t with respect to the Pavlov equation, then the scattering and the spectral data evolve in a simple way:

$$\begin{aligned}\sigma(\tau, \lambda, t) &= \sigma(\tau - \lambda^2 t, \lambda, 0), \\ \chi_{\pm}(\tau, \lambda, t) &= \chi_{\pm}(\tau - \lambda^2 t, \lambda, 0).\end{aligned}\tag{1.10}$$

The reconstruction of the potential consists of two steps:

1. One solves the following nonlinear integral equation for the time-dependent real Jost eigenfunction:

$$\psi_{-}(x, y, t, \lambda) - H_{\lambda} \chi_{-I}(\psi_{-}(x, y, t, \lambda), \lambda) + \chi_{-R}(\psi_{-}(x, y, t, \lambda), \lambda) = x - \lambda y - \lambda^2 t, \tag{1.11}$$

where χ_{-R} and χ_{-I} are the real and imaginary parts of χ_{-} , and H_{λ} is the Hilbert transform operator wrt λ

$$H_{\lambda} f(\lambda) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda - \lambda'} d\lambda'. \tag{1.12}$$

In [19] it is shown that, for Cauchy data satisfying some explicit small-norm conditions, equation (1.11) is uniquely solvable for all $t \geq 0$.

2. Once the real time-dependent Jost eigenfunction is known, the potential $v(x, y, t)$ is defined by:

$$v(x, y, t) = -\frac{1}{\pi} \int_{\mathbb{R}} \chi_{-I}(\psi_{-}(x, y, t, \lambda), \lambda) d\lambda. \tag{1.13}$$

In addition, in [19] it was shown that under the same analytic assumptions on the Cauchy data, the function $\omega(x, y, t, \lambda) = \psi_{-}(x, y, t, \lambda) - x + \lambda y + \lambda^2 t$ belongs to the spaces $L^{\infty}(d\lambda)$ and $L^2(d\lambda)$ for all real x, y and $t \geq 0$, and continuously depends on these variables. Moreover, for all $x, y \in \mathbb{R}, t \geq 0$, the following derivatives of ω :

$$\partial_x \omega, \quad \partial_y \omega, \quad \partial_t \omega, \quad \partial_x^2 \omega, \quad \partial_y^2 \omega, \quad \partial_x \partial_y \omega, \quad \partial_t \partial_x \omega,$$

are well-defined as elements of the space $L^2(d\lambda)$, they continuously depend on x, y, t and are uniformly bounded in $\mathbb{R} \times \mathbb{R} \times \overline{\mathbb{R}^+}$.

2 A lemma from integral geometry and its applications

How is it possible to guess the answer obtained in [20]? Consider the following approximate formula for reconstructing the function $v(x, y, t)$:

$$v(x, y, t) \sim -\frac{1}{\pi} \int_{\mathbb{R}} \chi_{-l}(\psi^{(0)}(x, y, t, \lambda), \lambda) d\lambda, \quad \text{where } \psi^{(0)}(x, y, t, \lambda) = x - \lambda y - \lambda^2 t. \quad (2.1)$$

In this approximation we replace the term $\omega(x, y, t, \lambda)$ by 0 in formula (1.13); this approximation is valid for $|x| \gg 1$, because in this region the wave function can be approximated by its normalization.

Let us apply the following result from integral geometry (the authors were not able to find this result in the literature).

Lemma 2.1 *Assume that we have a sufficiently good function $f(X, Y)$ in the real plane and a parabola $X = c_0 - c_2 Y^2$. Consider the following partial Radon data:*

$$I_l(a, b) = \int_{-\infty}^{+\infty} f(a + bY, Y) dY, \quad (2.2)$$

where only lines $X = d_0 + d_1 Y$, not intersecting the parabola $X = c_0 - c_2 Y^2$, are taken (but the lines are permitted to be tangent to parabola), see Figure 1. Then the integral

$$I_p(c_0, c_2) = \int_{-\infty}^{+\infty} f(c_0 - c_2 Y^2, Y) dY \quad (2.3)$$

can be expressed in terms of the data (2.2) by inverting the Melling transform (see formula (3.1)).

Assume now that $t > 0$ and $x > D_x$. Using the fact that $v(x, y, 0) = 0$ for $x > D_x$ we see that integrating $\chi_{-l}(\tau, \lambda)$ over any straight line not intersecting the parabola $x - \lambda y - \lambda^2 t$ is equal to zero (up to the above approximation error), see Figure 2. To construct $v(x, y, t)$ in our approximation, we have to calculate the integral of $\chi_{-l}(\tau, \lambda)$ on the parabola, which is 0 by Lemma 2.1, implying that v_t, v_x, v_y are also zero for large positive x and t positive. Therefore, from equation (1.2), it follows that $\partial_x^{-1} = -\int_x^{+\infty}$.

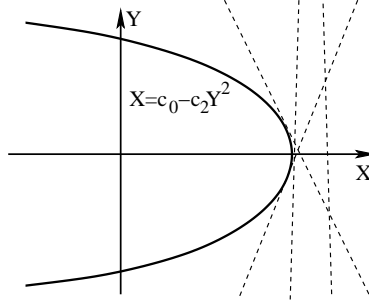


Figure 1: Integration contour for $I_p(c_0, c_2)$ (parabolic) and integration lines for $I_l(a, b)$ (dashed).

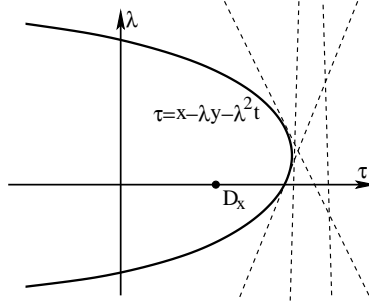


Figure 2: Integration contours for $t > 0$ (parabolic) and for $t = 0$ (dashed).

Now we would like to consider a completely different application of Lemma 2.1. Indeed a similar lemma, in which the parabola is replaced by the circle, is already known in the literature [21], and finds applications in tomography.

Assume that we have a two-dimensional linear tomographic problem with a convex opaque obstacle. In this situation, only the integrals along the straight lines non-intersecting the obstacle are known (the partial Radon data). If $f(X, Y)$ belongs to the Schwartz class, its reconstruction from the partial Radon data is unique, due to the hole theorem, but severely ill-posed [22]. In the class of potentials $f(X, Y)$ with polynomial decay, we do not have unique solvability of the inverse problem (the reconstruction of $f(X, Y)$) anymore. Consider, f.i., the function

$$f(X, Y) = \operatorname{Re} \left(\frac{1}{(X - X_0 + i(Y - Y_0))^n} \right), \quad n \geq 2,$$

where the point (X_0, Y_0) is located inside the obstacle. For all straight lines not intersecting the obstacle, the integral of $f(X, Y)$ along these lines is equal to 0 because $f(X, Y)$ is harmonic; moreover the function $f(X, Y)$ is well-localized for large n . It means that, if $f(X, Y)$ is a more general potential, its harmonic part does not contribute to the partial Radon data, i.e. it belongs to the kernel of the direct Radon transform [22]. Therefore, for the problem with obstacle, it is natural not to try to reconstruct the complete potential, but only some integral characteristics, like the integral over the boundary of the obstacle. The measure of the integral over the boundary of the obstacle should have the following property: for each function harmonic outside the obstacle with sufficient localization the integral should be equal to zero.

Let us discuss the cases of elliptic and parabolic obstacles, for which it is indeed possible to express the integral over the boundary of the obstacle in terms of the partial Radon data.

- The obstacle is bounded by the oval $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, i.e. the forbidden area is defined by the inequality $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \leq 1$.
- The obstacle is bounded by the parabola $x = -cy^2$, i.e. the forbidden area is: $x + cy^2 \leq 0$.

In the first case we would like to calculate the following integrals:

$$I_e(r) = \frac{1}{r} \int_{x^2/\alpha^2 + y^2/\beta^2 = r^2} f(x, y) \sqrt{\frac{dx^2}{\alpha^2} + \frac{dy^2}{\beta^2}}, \quad r \geq 1. \quad (2.4)$$

through the partial Radon data. Since, through the rescaling $\tilde{x} = x/\alpha$, $\tilde{y} = y/\beta$, the integral (2.4) can be reduced to the case of a circular obstacle:

$$I_e(r) = \frac{1}{r} \int_{\tilde{x}^2 + \tilde{y}^2 = r^2} f(\tilde{x}, \tilde{y}) ds, \quad ds^2 = d\tilde{x}^2 + d\tilde{y}^2, \quad r \geq 1, \quad (2.5)$$

we can use the following lemma from integral geometry [21]:

Lemma 2.2 *Assume that we have a sufficiently good function $f(x, y)$ in the real plane and the circle $x^2 + y^2 = r^2$. Then the integral*

$$I_c(r) = \frac{1}{r} \int_{x^2 + y^2 = r^2} f(x, y) ds, \quad ds^2 = dx^2 + dy^2, \quad (2.6)$$

can be expressed in terms of the partial radon data:

$$I_l(r_1, \phi) = \int_{l(r_1, \psi)} f(x, y) ds, \quad r_1 \geq r, \quad (2.7)$$

where $l(r_1, \phi)$ denotes the straight line such that r_1 is the distance between it and the origin, and ϕ denotes the angle between the line and the origin. Again, only the lines not intersecting the obstacle are considered.

In the second case the natural integral characteristics are:

$$I_p(c_0) = \int_{x+cy^2=c_0} f(x, y) dy, \quad c_0 > 0, \quad (2.8)$$

and we use Lemma 2.1.

We expect that Lemmas 2.1 and 2.2 can be generalized to a larger class of convex obstacles.

3 Proofs of the Lemmas

Proof of Lemma 2.1: Without loss of generality we may assume $c_2 = 1$. The line $x = a + by$ does not intersect the parabola iff $(a, b) \in U(c_0)$, where

$$U(c_0) = \left\{ (a, b) \mid a \geq \frac{b^2}{4} + c_0 \right\},$$

Let us calculate $J(c_0) = \iint_{U(c_0)} I_l(a, b) da db$.

$$\begin{aligned} J(c_0) &= \iint_{U(c_0)} I_l(a, b) da db = \iint_{U(c_0)} da db \int_{-\infty}^{+\infty} dy f(a + by, y) = \\ &= \int_{-\infty}^{+\infty} dy \iint_{a-c_0 \geq b^2/4} da db f(a + by, y) \end{aligned}$$

Denote $\tau = a + by$, $dadb = d\tau db$. then

$$J(c_0) = \int_{-\infty}^{+\infty} dy \iint_{\tau-c_0 \geq b^2/4+by} d\tau db f(\tau, y) =$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} dy \int_{\tau \geq c_0 - y^2} d\tau \int_{b^2/4 + by \leq \tau - c_0} db f(\tau, y) = \int_{-\infty}^{+\infty} dy \int_{\tau \geq c_0 - y^2} d\tau f(\tau, y) \int_{b^2/4 + by \leq \tau - c_0} 1 db = \\
&= \int_{-\infty}^{+\infty} dy \int_{\tau \geq c_0 - y^2} d\tau \left[f(\tau, y) 4 \sqrt{y^2 + \tau - c_0} \right].
\end{aligned}$$

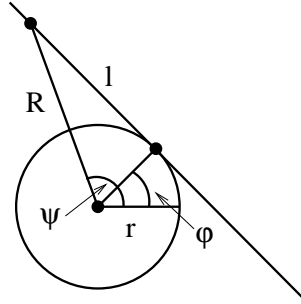
Denote $\tau = \tau' - y^2$; then

$$\begin{aligned}
J(c_0) &= 4 \int_{-\infty}^{+\infty} dy \int_{\tau' \geq c_0} d\tau' \left[f(\tau' - y^2, y) \sqrt{\tau' - c_0} \right] = 4 \int_{\tau' \geq c_0} d\tau' \sqrt{\tau' - c_0} \int_{-\infty}^{+\infty} dy f(\tau' - y^2, y) = \\
&= 4 \int_{\tau' \geq c_0} d\tau' \sqrt{\tau' - c_0} I_p(\tau');
\end{aligned} \tag{3.1}$$

Therefore our problem is reduced to the inversion of the Abel transform [23]:

$$I_p(\tau) = \frac{1}{2\pi} \frac{d}{d\tau} \int_{\tau}^{\infty} \frac{J'(c_0)}{\sqrt{c_0 - \tau}} dc_0. \tag{3.2}$$

Proof of Lemma 2.2:



Let us parametrize the lines in the plane by two parameters (r, ϕ) , where r denotes the distance from the origin and ϕ denotes the angle between the x -axis and the perpendicular from the origin to the line.

Denote by l the coordinate on the line, $l = 0$ at the perpendicular. Denote by (R, ψ) the polar coordinates of a point of the line. We have the following change

of coordinates:

$$\begin{cases} R = \sqrt{r^2 + l^2} \\ \psi = \phi + \arctan\left(\frac{l}{r}\right) \end{cases} \quad (3.3)$$

Assume that r is fixed.

$$d\psi \wedge dR = \det \begin{vmatrix} \frac{\partial \psi}{\partial \phi} & \frac{\partial \psi}{\partial R} \\ \frac{\partial \phi}{\partial \phi} & \frac{\partial \phi}{\partial R} \end{vmatrix} d\phi \wedge dR = \det \begin{vmatrix} 1 & \frac{r}{r^2 + l^2} \\ 0 & \frac{r}{\sqrt{l^2 + r^2}} \end{vmatrix} d\phi \wedge dl = \frac{l}{\sqrt{l^2 + r^2}} d\phi \wedge dl,$$

and

$$d\phi \wedge dl = \frac{R}{\sqrt{R^2 - r^2}} d\psi \wedge dR$$

Let us denote by $I_l(r, \phi)$ the integral

$$I_l(r, \phi) = \int_{-\infty}^{\infty} dl f(x(l, \phi), y(l, \phi)) dl$$

Consider the following integral:

$$\begin{aligned} J(r) &= \int_0^{2\pi} d\phi I_l(r, \phi) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dl f(x(l, \phi), y(l, \phi)) = \\ &= \int_0^{2\pi} d\psi \int_r^{\infty} dR f(R \cos \psi, R \sin \psi) \frac{R}{\sqrt{R^2 - r^2}} = \int_r^{\infty} I_c(R) \frac{R}{\sqrt{R^2 - r^2}} dR, \end{aligned}$$

where

$$I_c(R) = \int_0^{2\pi} f(R \cos \psi, R \sin \psi) d\psi.$$

Therefore our problem is reduced again to the inversion of the Abel transform:

$$I_c(R) = -\frac{2}{\pi} \int_R^{\infty} \frac{J'(r)}{\sqrt{r^2 - R^2}} dr. \quad (3.4)$$

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